PERGAMON
Topology 39 (2000) 573–587
TOPOLOGY
www.elsevier.com/locate/top

Lattice actions, 3-manifolds and homology

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Received 26 May 1998; received in revised form 22 January 1999; accepted 17 February 1999
Communicated by F.C. Kirwan

1. Introduction

A central question in the study of nonlinear versions of superrigidity (see [25]) is to determine all triples (Γ, M, ψ) where Γ is an irreducible lattice in a semisimple[2] Lie group of **R**-rank at least 2, M is a compact manifold of low enough dimension (for example $\leqslant n$ when $\Gamma = \mathrm{SL}(n, \mathbf{Z})$), and $\psi{:}\Gamma \to \mathrm{Diff}(M)$ is a smooth action with $\psi(\Gamma)$ infinite. While much is known about actions preserving a smooth volume density and connection, relatively little is known for general smooth actions.

The most basic and frequently studied example is the standard, linear action of $\mathrm{SL}(n, \mathbf{Z})$ on the n-torus T^n. A very crude property of this action is that the induced action on $H_1(T^n;\mathbf{Q})$ is infinite. Theorem I below asserts that any homologically infinite action of a higher-rank, irreducible lattice on a closed, irreducible 3-manifold is actually an action of a finite-index subgroup $\Gamma \leqslant \mathrm{SL}(3, \mathbf{Z})$ on the 3-torus T^3. Homologically infinite lattice actions on other (necessarily reducible) 3-manifolds do exist (see [14]); Theorems II and III below give strong restrictions on such actions.

Theorem I may be viewed as a topological analogue (in dimension three) of the theorem (see [27,6,24,4,5]) that if $\mathrm{SL}(n, \mathbf{Z})$, $n > 2$ admits a volume-preserving, connection-preserving, smooth action on a compact n-manifold M, then M is an affine torus.

To be more precise, an action $\psi : \Gamma \to \mathrm{Homeo}(M)$ of a group Γ on a compact n-manifold M induces an action on the rational homology of M:

$$\psi^i_* : \Gamma \to \mathrm{Aut}(H_i(M;\mathbf{Q})) = \mathrm{GL}(\mathrm{rank}(H_i(M;\mathbf{Q})), \mathbf{Q})$$

for each $0 \leqslant i \leqslant n$. If $\psi^i_*(\Gamma)$ is finite for each $0 \leqslant i \leqslant n$ we say that the action is *homologically finite*. An action which is not homologically finite is called *homologically infinite.*

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[1] Supported in part by NSF grant DMS 9704640, the second by NSF grant DMS 9626676.

[2] By a *semisimple* Lie group G we mean connected semisimple with finite center and no nontrivial compact factors.

PII: S0040-9383(99)00020-8

The statement of our first theorem will be for closed, irreducible 3-manifolds. Recall that a connected 3-manifold M is *irreducible* if every (tame) 2-sphere in M bounds a 3-ball. A classical theorem of Kneser states that every closed, orientable 3-manifold is a connected sum of irreducibles and copies of $S^2 \times S^1$.

Theorem I. *Let Γ be an irreducible lattice in a semisimple Lie group of* $\mathbf{R}$*-rank at least 2, and let M be a closed, orientable, irreducible, connected 3-manifold. If $\psi : \Gamma \to Homeo(M)$ is a homologically infinite action, then*

- *M is homeomorphic to T^3,*
- *Γ is a finite-index subgroup of (a conjugate of) $SL(3, \mathbf{Z})$, and*
- *ψ_* is conjugate to the restriction to Γ of the standard action of $SL(3, \mathbf{Z})$ on $H_1(T^3;Q)$. In fact:*
- *The action ψ is isotopically standard: for each element $\gamma \in \Gamma$, the homeomorphism $\psi(\gamma)$ is isotopic to the standard linear action of γ on T^3.*

In fact we will show that Theorem I holds more generally for actions of Γ on M by homotopy-equivalences.

Katok–Lewis [13] have constructed examples which show that Theorem I is false without the assumption that M is irreducible. Indeed there is a zoo of examples of homologically infinite actions of SL(3, $\mathbf{Z}$) on many different 3-manifolds (which by Theorem I are necessarily reducible).

The examples in [14] are constructed from the standard action of SL(3, $\mathbf{Z}$) on T^3. By blowing up the standard action at a global fixed point, one obtains a homologically infinite action of SL(3, $\mathbf{Z}$) on $T^3 \# \mathbf{RP}^3$. One may also blow up at every point of a finite orbit. By a gluing construction inspired both by [14] and by a different example due to Weinberger (explained in Section 3.1 below), one may construct examples of homologically infinite lattice actions on 3-manifolds with complicated topology, in particular with arbitrarily many T^3 connected summands. See Section 3.

The point of the following two theorems is that, at least on the level of homology, every homologically infinite lattice action on a closed 3-manifold is obtained by these gluing constructions.

The first of these theorems is best stated in terms of induced representations. Recall (see, e.g. [11]) that if Γ' is a subgroup of index d in a group Γ, and if $\rho' : \Gamma' \to \mathrm{GL}(V)$ is a representation of Γ' on an n-dimensional vector space V, then ρ' *induces* a representation

$$\rho : \Gamma \to \mathrm{GL}(W),$$

where W is a dn-dimensional vector space. Furthermore, W comes equipped with a direct sum decomposition

$$W = \bigoplus_{\alpha} V,$$

where α ranges over the d cosets of Γ' in Γ, and the restriction of ρ to Γ' is a direct sum of d representations equivalent to ρ'.

Theorem II. *Let Γ be an irreducible, nonuniform lattice in a semisimple Lie group of* $\mathbf{R}$*-rank at least 2, and let M be any closed, orientable, connected 3-manifold. Suppose $\psi : \Gamma \to Homeo(M)$ is* *a homologically infinite action. Then*

- *Γ is commensurable with (a conjugate of) $SL(3, \mathbf{Z})$,*
- *M has a connected summand N which is itself is a connected sum of 3-tori*

$$N = \#_{i=1}^{r} T_i^3$$

  for some $r \geqslant 1$,
- *The action of Γ on $H_1(M; \mathbf{Q})$ is a direct sum of representations $\rho_1 \oplus \rho_2$ where*
  - *ρ_2 is a finite representation of Γ,*
  - *ρ_1 is a representation of Γ on the direct summand $H_1(N; \mathbf{Q})$ of $H_1(M; \mathbf{Q})$,*
  - *ρ_1 is a direct sum of representations of Γ induced by standard three-dimensional representations of finite-index subgroups of Γ, and*
  - *The decomposition of $H_1(N;\mathbf{Q})$ into three-dimensional subspaces associated with the description of ρ_1 as a sum of induced representations coincides with the topologically defined decomposition*

$$H_1(N; \mathbf{Q}) = \bigoplus_{i=1}^{r} H_1(T_i^3; \mathbf{Q})$$

In Section 3 we show how induced representations actually arise in examples of SL(3, **Z**)-actions on closed 3-manifolds.

An ingredient in our proof of Theorem II is the fact (see [1]) that any homomorphism of a nonuniform Γ into the outer automorphism group $\mathrm{Out}(F_n)$ of a finitely generated free group must have finite image. We conjecture that this result, and therefore a version of Theorem II for uniform lattices, should also hold.

Homomorphisms of Γ into $\mathrm{Out}(F_n)$ arise precisely in the cases where M^3 has $n \geqslant 1$ connected summands homeomorphic to $S^2 \times S^1$. In the case when M is assumed to have no such summands, we are able to obtain the desired result.

Theorem III. *Let Γ be a irreducible, uniform lattice in a semisimple Lie group of* **R***-rank at least 2, and let M be any closed, orientable, connected 3-manifold with no $S^2 \times S^1$ connected summands. Then every action $\psi{:}\Gamma \to Homeo(M)$ is homologically finite.*

Note that there are many homologically finite lattice actions $\psi : \Gamma \to \mathrm{Homeo}(M)$ with $\psi(\Gamma)$ infinite. For example lattices in the semisimple Lie groups SL(4, **R**), SP(4, **R**), and O(4, 1) all admit such actions on S^3.

2. The irreducible case

This section is devoted to a proof of Theorem I.

2.1. Rational homology 3-spheres and Haken 3-manifolds

A 2-sided, closed surface $\Sigma \neq S^2$ in M is *incompressible* if the inclusion $\Sigma \hookrightarrow M$ induces an injection $\pi_1(\Sigma) \to \pi_1(M)$. The 3-manifold M is *Haken* if it contains a 2-sided, closed, incompressible surface. It is a standard fact (see, e.g. [9, Lemma III.10]) that a closed 3-manifold M is Haken if $H_1(M; \mathbf{Q}) \neq 0$. By Poincaré duality, if M is not Haken then M is a *rational homology 3-sphere*, that is M has the same rational homology as S^3.

Since any action on a rational homology 3-sphere is clearly homologically finite, it suffices to prove Theorem I for closed Haken 3-manifolds.

2.2. Mapping class groups

The *mapping class group* of a compact manifold M, denoted Mod(M), is the group π_0(Homeo(M)) of homeomorphisms of M modulo the subgroup of homeomorphisms which are isotopic to the identity. The index 2 subgroup of Mod(M) consisting of orientation-preserving mapping classes will be denoted $\mathrm{Mod}^+(M)$.

A result of Waldhausen [22] states that, for Haken 3-manifolds M, the group Mod(M) is isomorphic to the group of self-homotopy equivalences of M up to homotopy.

2.2.1. Relative mapping class groups

We will also need to consider certain "relative mapping class groups", defined as follows. Let $A_1, \ldots, A_r$ be a collection of possibly empty subspaces of M. We denote by $\mathrm{Mod}\,(M, A_1, \ldots, A_r)$ the group of path components of the group

$$\mathrm{Homeo}(M, A_1, \ldots, A_r) = \{h \in \mathrm{Homeo}(M) : h(A_i) = A_i \text{ for all } 1 \leqslant i \leqslant r\}.$$

2.2.2. Dehn twists in 3-manifolds

Let T be an embedded torus in M. A *Dehn twist* about T is a homeomorphism t of M which can be obtained from the following construction. Choose a product neighborhood $T \times I$ of T in M. Choose a homeomorphic identification $h: S^1 \times S^1 \to T$, where S^1 is the unit circle in the complex plane, and let

$$t(h(z, w), s) = (h(\mathrm{e}^{2\pi \mathrm{i} s} z, w), s)$$

and extend t to be the identity on $M - (T \times I)$. The element $[t] \in \mathrm{Mod}(M)$ depends only on the isotopy class of $h(S^1 \times \{1\})$.

2.3. Representations into mapping class groups

The main ingredient in the proof of Theorem I is Theorem 2.2 below, which concerns representations of lattices into mapping class groups of Haken 3-manifolds. The proof of Theorem 2.2 depends in turn on the following analogous result for mapping class groups of surfaces.

Theorem 2.1 (Actions on surfaces)**.** *Let Γ be an irreducible lattice in a semisimple lie group G of $\mathbf{R}$-rank at least two, and let S be a compact surface. Then any homomorphism $\phi : \Gamma \to Mod(S)$ has finite image.*

A proof of Theorem 2.1 is given in [3].

An action $\psi : \Gamma \to \mathrm{Homeo}(M)$ of a group Γ on a manifold M is *virtually isotopically trivial* if there is a finite-index subgroup $H < \Gamma$ such that $\psi(h)$ is isotopic to the identity for each $h \in H$.

Theorem 2.2 (Actions on Haken 3-manifolds). *Let Γ be an irreducible lattice in a semisimple Lie group of* **R***-rank at least 2, and let M be a closed, Haken 3-manifold. Suppose M is not homeomorphic to T^3. Then any homomorphism $\Gamma \to Mod(M)$ has finite image. In particular, every topological action of Γ on M is virtually isotopically trivial, hence homologically finite.*

Subsections 2.4–2.7 will be devoted to proving Theorem 2.2. The proof that Theorem 2.2 implies Theorem I will be given in Section 2.8.

2.4. Thin groups

The proof of Theorem 2.2 will be made simpler by abstracting the situation.

Definition (*Thin group*). We say that Γ is a *higher-rank lattice* if Γ is an irreducible lattice in a semisimple Lie group of **R**-rank at least 2. We say that a group N is *thin* if it has the following property: any homomorphism $\Gamma \to N$ of a higher-rank lattice Γ into N must have finite image.

Remark. The Margulis Finiteness Theorem (see, e.g. [26, Theorem 8.1]) says that any normal subgroup of a higher-rank lattice Γ must be finite or of finite index. Hence if N is not thin, then the kernel K of some homomorphism $f: \Gamma \to N$ is finite. As Γ is residually finite, there is a finite-index subgroup Λ of Γ which injects into N. It follows that a group N is thin if and only if it does not contain a subgroup H isomorphic to a higher-rank lattice.

Examples:

1. Amenable groups are thin. This follows from the fact that higher-rank lattices have Kazhdan's property T, together with the fact that any homomorphism of a group with property T into an amenable group has precompact image (see Ch. 7 of [26]). In particular,
2. Finite groups and abelian groups are thin.
3. Let G be any group which has the property that any infinite, finitely generated subgroup of G has infinite abelianization. Then the fact that higher-rank lattices have property T shows that G is thin. In particular free groups are thin.
4. Theorem 2.1 shows that $\mathrm{Mod}(S)$ is thin for any compact surface S.
5. Subgroups of thin groups are thin.

Lemma 2.3 (Extensions). *Let*

$$1 \to A \xrightarrow{i} B \xrightarrow{p} C \to 1$$

be an exact sequence of groups. If A and C are thin then B is thin. In particular products of thin groups are thin.

Proof. Let $\psi : \Gamma \to B$ be a homomorphism from a higher-rank lattice Γ. Since C is thin, $p\circ\psi(\Gamma)$ is finite. Since Γ is residually finite, it follows that there is a finite-index subgroup Λ of Γ with

$$\psi(\Lambda) \subseteq \ker p \subseteq i(A).$$

But A is thin so $\psi(\Lambda)$ is finite. $\square$

2.5. Seifert fibered spaces

An important special case in our considerations will be 3-manifolds which are *Seifert fibered.* The theory of Seifert fibered spaces may be interpreted as the theory of 3-manifolds admitting effective S^1 actions. There is an extensive literature on Seifert fibered spaces, see for example [7, 9, 12, 10, 20, 21].

If M is a compact 3-manifold with a given S^1 action, it can be shown that

- The orbit space of the S^1 action on M is a compact surface Σ, called the *base surface*.
- There are finitely many orbits for which the action of S^1 is not free. These are called *exceptional orbits*.

Thus an S^1 action on M determines a map f from M to a compact surface Σ and a finite set of points $X \subset \Sigma$ (corresponding to the exceptional orbits). A triple (Σ, f, X) is called a *Seifert fibration* of M if it arises via this construction from an S^1 action on M. By a *Seifert fibered space* we mean a compact 3-manifold equipped with a Seifert fibration. This is equivalent to the more direct topological definition, for which we refer the reader to the references mentioned above.

The inverse images under f of points of Σ are called the *fibers* of the Seifert fibration. If we think of the Seifert fibration as coming from an S^1 action, the fibers are just the orbits. A homeomorphism of a Seifert fibered space M is *fiber-preserving* if it maps each fiber to a fiber.

Proposition 2.4 (*Seifert fibered case*)**.** *Let M be a compact, Haken, Seifert fibered 3-manifold. Then either $Mod(M)$ is thin or $M = T^3$ and $Mod(M) = GL(3, \mathbf{Z})$.*

Proof. First assume that M is not homeomorphic to one of the six exceptional 3-manifolds listed below; the theorems we now quote apply to any Haken, Seifert fibered 3-manifold which is not homeomorphic to one of these six exceptions. The theorems are due to Mccullough [19], building on work of Johannson [12]. We remark that in applying the theorems in [19], when M has nonempty boundary we will take the "boundary patterns" m and m_1 to be ∂M.

It follows from [19, Theorem 3.5.1] (see also the discussion at the start of the proof of Theorem 3.6.1 of [19] that the group $\mathrm{Mod}^+(M)$ is isomorphic to the group $\mathrm{Mod}_f^+(M)$ of orientation-preserving mapping classes which are represented by fiber-preserving homeomorphisms.

Let Σ denote the base surface of M. By [19, Theorem 3.5.2 and Lemma 3.5.3] (see also Propositions 25.2 and 25.3 of [12]), there is an exact sequence

$$1 \to H_1(\Sigma, \partial\Sigma) \to \mathrm{Mod}_f^+(M) \to \mathrm{Mod}^*(\Sigma) \to 1,$$

where Mod*(Σ) is a finite index subgroup of Mod(Σ'), where Σ' is the surface obtained from Σ by deleting the points corresponding to exceptional orbits of M. (The kernel $H_1(\Sigma, \partial\Sigma)$ can be thought of as the group of "vertical mapping classes"; i.e. those represented by homeomorphisms that map each fiber to itself. For example, a simple closed curve γ in Σ, thought of as an element of $H_1(\Sigma, \partial\Sigma)$, lifts to a torus in M, and a Dehn twist around about this torus gives an element of $\mathrm{Mod}_f^+(M)$ projecting to the trivial element of Mod*(Σ).)

Theorem 2.1 shows that Mod(Σ') (hence Mod*(Σ)) is thin. Since $H_1(\Sigma, \partial\Sigma)$ is abelian, it follows easily from Lemma 2.3 that $\mathrm{Mod}_f^+(M)$, hence $\mathrm{Mod}^+(M)$, hence Mod(M), is thin.

We now list the six exceptional homeomorphism types of Seifert-fibered spaces. The computation of mapping class groups in the first five cases is given in section 3.4 of [19], the sixth is given in [2]. We claim that, for these exceptional Seifert-fibered spaces M, either Mod(M) is thin or $M = T^3$ and Mod(M) is isomorphic to GL(3, **Z**). This will complete the proof of the proposition.

We remark that we are cataloguing the exceptions only up to homeomorphism; some of these manifolds admit two or more inequivalent Seifert fibrations.

1. M is homeomorphic to $T^2 \times [0, 1]$. In this case Mod(M) has a subgroup of index 2 which is isomorphic to GL(2, **Z**). Hence Mod(M) is thin.
2. M is homeomorphic to a twisted I-bundle over a Klein bottle. In this case Mod(M) is finite, hence thin.
3. M is an S^1-bundle over the torus. If the Euler class of this bundle is 0 then Mod(M) is isomorphic to GL(3, **Z**). If the Euler class of this bundle is not 0 then Mod(M) is virtually free, hence thin.
4. M is an S^1-bundle over the Klein bottle. If the Euler class of this bundle is 0 then Mod(M) is virtually free, hence thin. If the Euler class is not 0 then Mod(M) is finite.
5. M fibers over S^2 with three exceptional orbits. In this case Mod(M) is finite.
6. M is a certain closed, flat 3-manifold called the *Hantsche-Wendt manifold* ([20, p.133, p.138, 2, pp. 478–481)]. Its Seifert invariants are $\{-1;(n_2,1);(2,1),(2,1)\}$. In this case Mod($M$) is finite. □

2.6. The characteristic submanifold

In this brief subsection we review the characteristic submanifold theory (see [10,12]).

Let M be a closed, orientable, Haken 3-manifold. It is shown in [10,12] that there exists a compact submanifold Σ of M with the following properties:

(1) every connected component of Σ is a Seifert fibered space,
(2) every component of $\partial\Sigma$ is an incompressible torus in M,
(3) every Seifert fibered submanifold of M whose boundary components are incompressible tori is isotopic to a submanifold of Σ.

We can clearly choose Σ so that in addition:
(4) no union of a proper subset of components of Σ satisfies property (3).

It is also shown in [10,12] that a submanifold satisfying (1)–(4) is unique up to ambient isotopy. The submanifold Σ is called the *characteristic submanifold* of M. Denote by $\Sigma_1, \ldots, \Sigma_n$ be the connected components of the characteristic submanifold.

2.7. Proof of Theorem 2.2

If $\Sigma = \emptyset$ then ([12, Section 27]) Mod(M) is finite, hence thin. If $\Sigma = M$ then M is Seifert-fibered, so that either M is homeomorphic to T^3 or Mod(M) is thin by Section 2.5. So to prove Theorem 2.2, it is enough to assume that Σ is nonempty and not all of M. For the rest of this section we assume this.

Based on [12], Proposition 4.1.1 of [19] states that if M is not a torus bundle over the circle then the natural homomorphism

$$\mathrm{Mod}\,(M, \Sigma) \to \mathrm{Mod}\,(M)$$

is an isomorphism. The surjectivity of this homomorphism is essentially the uniqueness of the characteristic submanifold up to ambient isotopy.

Now no component of Σ can be a torus bundle over S^1 unless M itself is a torus bundle. Since M is not a Seifert fibered space (by assumption), it follows that in this case M admits a metric locally modelled on the three-dimensional geometry Sol (see [21]). It is shown in Proposition 4.1.2 of [21] that torus bundles M over the circle which admit a *Sol* metric have Mod(M) finite, hence thin, so we henceforth disregard this case.

Hence it suffices to prove that Mod(M, Σ) is thin. Clearly, the natural inclusion

$$\mathrm{Mod}\,(M, \Sigma_1, \ldots, \Sigma_n) \to \mathrm{Mod}\,(M, \Sigma)$$

has finite-index image. So it suffices to prove that Mod($M, \Sigma_1, \ldots, \Sigma_n$) is thin.

There is an obvious homomorphism

$$\phi\colon \mathrm{Mod}(M, \Sigma_1, \ldots, \Sigma_n) \to \mathrm{Mod}(\overline{M - \Sigma}, \partial\Sigma),$$

which by [19, Lemma 4.2.1] has finite image. (The point is that 3-manifolds M for which every incompressible torus in M is parallel into ∂M have finite Mod(M) ([12, Section 27]). This may be viewed as a generalization of the theorem that compact, hyperbolic 3-manifolds have finite mapping class groups.)

This shows that we need only show that the kernel of ϕ, which we shall denote by $\mathrm{Mod}_{\mathrm{Ker}}(M, \Sigma_1, \ldots, \Sigma_n)$, is thin.

Define

$$\mathrm{Twist}\,(\partial\Sigma) \subset \mathrm{Mod}_{\mathrm{Ker}}(M, \Sigma_1, \ldots, \Sigma_n)$$

to be the group generated by all Dehn twists about all components of $\partial\Sigma$. Then Twist ($\partial\Sigma$) is readily seen to be a finitely-generated abelian group.

Now every element $h \in \mathrm{Mod}_{\mathrm{Ker}}(M, \Sigma_1, \ldots, \Sigma_n)$leaves each Σ_i invariant, and $h|_{\partial\Sigma_i}$ is isotopic to the identity. Hence each fiber of the Seifert-fibered space Σ_i is homotopic to its image under h. This implies (see [12]) that the restriction of h to each Σ_i is homotopic to a fiber-preserving homeomorphism. So there is a homomorphism

$$\mathrm{Mod}_{\mathrm{Ker}}(M, \Sigma_1, \ldots, \Sigma_n) \to \mathrm{Mod}_{\mathrm{f}}(\Sigma_i; \partial\Sigma_i)$$

for each $1 \leqslant i \leqslant n$, where $\mathrm{Mod}_{\mathrm{f}}(\Sigma_i; \partial\Sigma_i)$ denotes the normal subgroup of $\mathrm{Mod}_{\mathrm{f}}(\Sigma_i)$ consisting of fiber-preserving mapping classes $[f]$ for which the restriction of f to $\partial\Sigma_i$ is isotopic to the identity.

Hence there is a natural homomorphism

$$\mathrm{Mod}_{\mathrm{Ker}}(M, \Sigma_1, \ldots, \Sigma_n) \to \prod_{i=1}^{n} \mathrm{Mod}_{\mathrm{f}}(\Sigma_i; \partial\Sigma_i).$$

Lemma 4.2.2 of [19] shows that the kernel of this homomorphism is Twist$(\partial\Sigma)$; that is, the sequence

$$1 \to \mathrm{Twist}\,(\partial\Sigma) \to \mathrm{Mod}_{\mathrm{Ker}}(M, \Sigma_1, \ldots, \Sigma_n) \to \prod_{i=1}^{n} \mathrm{Mod}_{\mathrm{f}}(\Sigma_i; \partial\Sigma_i) \to 1 \qquad (*)$$

is exact.

Since M is not Seifert fibered, each Σ_i has nonempty boundary. In particular, no Σ_i is homeomorphic to T^3. Hence by Section 2.5 $\mathrm{Mod}(\Sigma_i)$ is thin. In particular each $\mathrm{Mod}_{\mathrm{f}}(\Sigma_i; \partial\Sigma_i) \subseteq \mathrm{Mod}(\Sigma_i)$ is thin. By Lemma 2.3 the product of these groups is thin. Since Twist$(\partial\Sigma)$ is abelian it is thin. An application of Lemma 2.3 to the exact sequence $(*)$ now gives that $\mathrm{Mod}_{\mathrm{Ker}}(M, \Sigma_1, \ldots, \Sigma_n)$ is thin, and we are done. $\square$

2.8. Finishing the proof of Theorem I

Theorem I follows from Theorem 2.2 as follows If $\psi : \Gamma \to \mathrm{Homeo}(M)$ is a homologically infinite action on an irreducible M, it must be that M is Haken by Section 2.1. If M is not homeomorphic to T^3, then by Theorem 2.2 it follows that the composition

$$\Gamma \xrightarrow{\psi} \mathrm{Homeo}(M) \xrightarrow{\pi} \mathrm{Mod}(M)$$

has finite image (here π is the natural projection). In particular, the action ψ is homologically finite, a contradiction. Hence M is homeomorphic to T^3.

The action $\psi_* : \Gamma \to \mathrm{Aut}(H_1(T^3, \mathbf{Z})) = \mathrm{GL}(3, \mathbf{Z})$ has infinite image since ψ is homologically infinite. We regard ψ_* as a representation of Γ into $\mathrm{GL}(3, \mathbf{R})$. Since $\psi_*(\Gamma)$ is an infinite subgroup of $\mathrm{GL}(3, \mathbf{Z})$, it is clear that $\psi_*(\Gamma)$ does not have compact closure. So we may apply Margulis's Superrigidity Theorem (see, e.g., [26, Theorem 5.1.2]), which gives that the representation ψ_* extends to a representation $\rho: G \to \mathrm{GL}(3, \mathbf{R})$, where G is the semisimple Lie group of $\mathbf{R}$-rank at least 2 in which Γ is a lattice.

First consider the case where G is simple. Then since G has $\mathbf{R}$-rank at least 2 and ρ is nontrivial, it follows easily that $G = \mathrm{SL}(3, \mathbf{R})$ and that ρ is an isomorphism of G onto $\mathrm{SL}(3, \mathbf{R})$. Hence ρ maps Γ isomorphically to a subgroup of $\mathrm{SL}(3, \mathbf{Z})$. Since Γ is a lattice in $\mathrm{SL}(3, \mathbf{R})$ which is contained in $\mathrm{SL}(3, \mathbf{Z})$, Γ must have finite index in $\mathrm{SL}(3, \mathbf{Z})$.

The third claim of Theorem I is now clear. The fourth claim follows from the third claim and the fact that the natural map

$$\mathrm{Mod}(T^3) \to \mathrm{Aut}(H_1(T^3; \mathbf{Z})) \approx \mathrm{GL}(3, \mathbf{Z})$$

is an isomorphism.

Now, if G were not simple then ρ would be projection to an $\mathrm{SL}(3, \mathbf{R})$ factor. Since Γ is irreducible the image $\rho(\Gamma)$ would be dense in $\mathrm{SL}(3, \mathbf{R})$. But this cannot happen since $\rho(\Gamma) \subseteq \mathrm{SL}(3, \mathbf{Z})$. Hence it must be that G is simple. $\square$

3. Constructing homologically infinite actions

In this section we give some examples inspired from those of Katok–Lewis [13,14] and Weinberger [23] which show that Theorem I is false without the assumption that M is irreducible. These examples also illustrate the phenomenon described in the conclusion of Theorem II.

Let $X = \{x_1, \dots, x_n\}$ be a finite subset of T^3 which is invariant under the standard linear action ρ of $\mathrm{SL}(3, \mathbf{Z})$ on $T^3 = R^3/Z^3$. Many such finite sets X exist. Indeed, for any positive integer m, $\mathrm{SL}(3, \mathbf{Z})$ permutes the m^3 distinct points $(p_1/m, p_2/m, p_3/m) \in T^3$, where $0 \leqslant p_i < m$.

Choose a disjoint collection of closed balls $C_1, \dots, C_n$ about $x_1, \dots, x_n$, where each C_i is the image of a round ball in $\mathbf{R}^3$ whose center maps to x_i. We identify each C_i with the corresponding round ball in $\mathbf{R}^3$ via the restriction of the covering projection. For each i we choose a round ball B_i centered at x_i and contained in the interior of C_i.

Let $N = T^3 \times \{0, \dots, n\}$, and define a representation σ of $\mathrm{SL}(3, \mathbf{Z})$ on the group of permutations of $\{0, \dots, n\}$ by setting $\sigma(\gamma)(0) = 0$, and setting $\sigma(\gamma)(i) = j$ if $\rho(\gamma)(x_i) = x_j$. We define an action ρ' of $\mathrm{SL}(3, \mathbf{Z})$ on N by

$$\rho'(\gamma)(x, i) = (\rho(x), \sigma(i))$$

for $x \in T^3$, $i \in \{0, \dots, n\}$. Note that the finite set

$$F = (X \times \{0\}) \cup \{(x_i, i)\}_{i=1}^{n} \subset N$$

is invariant under $\rho'(\mathrm{SL}(3, \mathbf{Z}))$. Let

$$B = \bigcup_{i=1}^{n} (B_i \times \{0, i\}).$$

For each $i = 1, \dots, n$ we choose a homeomorphism $g_i : C_i - \{x_i\} \to C_i - B_i$ which maps each radius into itself and is the identity on ∂C_i. The g_i's determine a homeomorphism $f : N - F \to N - B$ (where f is the identity on $N - \bigcup_{i=1}^{n}(C_i \times \{0, i\})$ and is $g_i \times \mathrm{Id}$ on $C_i \times \{0, i\}$.)

We now use f to conjugate the action of $\mathrm{SL}(3, \mathbf{Z})$ on $N - F$ to an action ρ'' of $\mathrm{SL}(3, \mathbf{Z})$ on $N - B$. We claim that we can extend ρ'' to an action of $\mathrm{SL}(3, \mathbf{Z})$ on the manifold with boundary $N' = N - \mathrm{int}(B)$. To see this, note that $\partial N'$ has a collar neighborhood in N' which can be identified with $\partial N' \times [0, 1]$ in such a way that each $(x, j) \times [0, 1]$ is the intersection of a radius of some $C_i \times \{j\}$ with $B_i \times \{j\}$ where $j = 0$ or $j = i$. As usual with boundary collars, we identify (x, j) with $((x, j), 0)$. Since the original action is linear and each g_j is radial, we have that for each $\gamma \in \mathrm{SL}(3, \mathbf{Z})$ and each (x, j) there is an $\varepsilon > 0$ such that $\gamma \cdot ((x, j) \times (0, \varepsilon))$ is contained in $(x', j') \times [0, 1]$ for some (x', j'). The claim clearly follows.

The extension of ρ'' to N' will also be denoted ρ''.

Note that N' is a compact manifold with $n + 1$ connected components and $2n$ boundary components, each of which is homeomorphic to a 2-sphere.

We now form a closed, connected 3-manifold M as a quotient of N' by identifying $\partial B_i \times \{0\}$ with $\partial B_i \times \{i\}$ for each $i = 1, \dots, n$. Clearly, the action ρ'' descends to an action $\psi : \mathrm{SL}(3, \mathbf{Z}) \to \mathrm{Homeo}\,(M)$. This is the desired action. Note that M is a connected sum of 3-tori.

What does the representation $\psi_* : \mathrm{SL}(3, \mathbf{Z}) \to \mathrm{Aut}(H_1(M; \mathbf{Q}))$ look like? In the special case when X is a finite $\mathrm{SL}(3, \mathbf{Z})$-orbit of a single-point $x \in T^3$ under the standard linear action, ψ_* is a direct sum of the standard representation of $\mathrm{SL}(3, \mathbf{Z})$ with a representation of $\mathrm{SL}(3, \mathbf{Z})$ induced from the standard representation of a finite-index subgroup $\Gamma \subseteq \mathrm{SL}(3,\mathbf{Z})$, where Γ is the stabilizer of x under the standard action. Note that in the case where x is the global fixed-point for the standard action of $\mathrm{SL}(3,\mathbf{Z})$, the action ψ is the "double" of the standard action.

When X is a union of $r>0$ finite $\mathrm{SL}(3,\mathbf{Z})$-orbits of points $y_1, \dots, y_r$, the representation ψ_* is a direct sum of the standard representation with the direct sum of r induced representations of finite-index subgroups of $\mathrm{SL}(3,\mathbf{Z})$ corresponding to the stabilizers of the y_i for $i=1, \dots, r$.

In [14], homologically infinite actions of $\mathrm{SL}(3,\mathbf{Z})$ on $T^3 \# \mathbf{RP}^3$ are constructed which are real-analytic and volume-preserving. It would be interesting to determine whether or not there are actions as above with these properties.

3.1. *Exotic* SL(3,Z) *actions on* T^3

In this subsection we describe actions of $\mathrm{SL}(3,\mathbf{Z})$ on T^3 which are not topologically conjugate to the standard linear action. The construction of these actions is due to Shmuel Weinberger (and in fact inspired the construction above).

Using the same construction as above, we can construct an action ψ of $\mathrm{SL}(3,\mathbf{Z})$ on T^3 minus the interior of a single ball B_1, which is a compact 3-manifold with boundary a 2-sphere ∂B_1. It is not hard to check that we can extend ψ to an action of $\mathrm{SL}(3,\mathbf{Z})$ on all of T^3 by either:

(1) coning off the action of each $\psi(\gamma)$ on ∂B_1 to a homeomorphism of B_1 (the "Alexander trick"), or
(2) gluing back in to $T^3 - \mathrm{int}(B_1)$ the standard action of $\mathrm{SL}(3,\mathbf{Z})$ on B_1 via the linear action on the space of rays through the origin in $\mathbf{R}^3$.

In either case we obtain an action ψ of $\mathrm{SL}(3,\mathbf{Z})$ on T^3 by homeomorphisms which leave invariant the 2-sphere ∂B_1. Since ψ leaves the nonempty open sets $\mathrm{int}(B_1)$ and B_1^c invariant, and since the standard action of $\mathrm{SL}(3,\mathbf{Z})$ on T^3 is ergodic, it follows that ψ is not topologically conjugate to the standard action. However, it is easy to check that the action ψ is isotopically standard.

By iterations and variations of the above procedure, it is not hard to see that there are uncountably many actions of $\mathrm{SL}(3,\mathbf{Z})$ on T^3, no two of which are topologically conjugate.

4. Proof of Theorems II and III

We begin by proving Theorem II.

Since M is orientable, any connected summand of M with infinite-cyclic fundamental group must be homeomorphic to $S^2 \times S^1$ (see [7]). By Kneser's Theorem (see [7]) we may write M as a finite connected sum

$$M = M_1 \# \cdots \# M_n \# B \# C,$$

where B is a connected sum of $m \geqslant 0$ copies of $S^2 \times S^1$, C has trivial fundamental group, and each M_i is irreducible and has nontrivial fundamental group. Setting $A_i = \pi_1(M_i)$ we can therefore write

$\pi_1(M)$ as a free product

$$\pi_1(M) = A_1 * \cdots * A_n * F_m,$$

where $F_m = \pi_1(B)$ is a free group of rank $m \geqslant 0$, and the groups $A_i = \pi_1(M_i)$ are freely indecomposable, nontrivial, and not infinite cyclic.

The action $\psi : \Gamma \to \mathrm{Homeo}\,(M)$ induces a homomorphism

$$\Psi : \Gamma \to \mathrm{Out}(\pi_1(M)).$$

Let $\phi \in \mathrm{Aut}(\pi_1(M))$ be given. Since each A_i is nontrivial, not infinite cyclic, and freely indecomposable, the Kurosh Subgroup shows that for each i there exist an integer $\sigma(i)$ with $1 \leqslant \sigma(i) \leqslant n$ and an element $g_i \in \pi_1(M)$ with

$$\phi(A_i) \subseteq g_i A_{\sigma(i)} g_i^{-1}.$$

Since ϕ is surjective it follows easily that $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$ is a permutation. The surjectivity of ϕ also implies that

$$\phi(A_i) = g_i A_{\sigma(i)} g_i^{-1}$$

for each i. Note that σ depends only on the equivalence class $[\phi] \in \mathrm{Out}(\pi_1(M))$.

This gives a permutation representation $\omega : \Gamma \to S_n$. Let Γ_i denote the pre-image under ω of the stabilizer of i in S_n. Then Ψ gives rise to a representation

$$\xi_i : \Gamma_i \to \mathrm{Out}(A_i).$$

According to a Theorem of Waldhausen [22], the natural homomorphism $\mathrm{Mod}(M_i) \to \mathrm{Out}(A_i)$ is an isomorphism. Since Γ_i is a higher-rank lattice, we may apply Theorem 2.2 to conclude that for each i, either $\xi_i(\Gamma_i)$ is finite or M_i is homeomorphic to T^3.

Under the action of Γ on $H_1(M; \mathbf{Q})$, each $\gamma \in \Gamma$ maps $H_1(M_i; \mathbf{Q})$ isomorphically onto $H_1(M_{\omega(\gamma)(i)}; \mathbf{Q})$ for $1 \leqslant i \leqslant n$. Let

$$U = \bigoplus_{i=1}^{n} H_1(M_i; \mathbf{Q})$$

and $\{i_1, \ldots, i_k\}$ be a complete set of orbit representatives for the permutation representation ω. Let η_j be the representation of Γ_{i_j} on $H_1(M_{i_j}; \mathbf{Q})$ coming from the representation ξ_{i_j}.

Then U is Γ-invariant and the representation

$$\nu_1 : \Gamma \to \mathrm{Aut}(U)$$

is a direct sum of representations of Γ induced from $\eta_1, \ldots, \eta_k$.

Since U is Γ-invariant, the representation $\psi_* : \Gamma \to \mathrm{Aut}(H_1(M; \mathbf{Q}))$ defines a representation $\nu_2 : \Gamma \to \mathrm{Aut}(V)$ where we define V to be $H_1(M; \mathbf{Q})/U$. We claim that the short exact sequence

$$1 \to U \to H_1(M; Q) \to V \to 1$$

splits as an exact sequence of $\mathbf{Z}[\Gamma]$-modules. To see this it is enough to show that $\mathrm{Ext}_G(U, V)$ is trivial (see [11, Theorem 6.7]). But there is (see [11, Theorem 6.12]) an isomorphism

$$\mathrm{Ext}_G(U, V) \approx H^1(\Gamma; \mathrm{Hom}(V, U)),$$

where Hom(V,U) is considered as a $\mathbf{Z}[\Gamma]$-module via the diagonal action of Γ. A theorem of Margulis (see [17, IX.5.9]) states that $H^1(\Gamma, \rho) = 0$ for any higher-rank lattice Γ and any finite-dimensional representation ρ (the theorem applies to these lattices by Remark (i) on p.289 of [17]). Hence $H^1(\Gamma, \mathrm{Hom}(V, U)) = 0$ and the claim is proved. Thus we may regard ψ_* as a direct sum $v_1 \oplus v_2$.

We claim that the representation v_2 is finite. To see this, note that the normal closure $\langle\langle A_1 * \cdots * A_n \rangle\rangle$ of $A_1 * \cdots * A_n$ is characteristic by the discussion above. Hence there is a natural homomorphism $\mathrm{Out}(\pi_1(M)) \to \mathrm{Out}(\pi_1(M))/\langle\langle A_1 * \cdots * A_n \rangle\rangle)$. Consider the composition

$$\Gamma \to \mathrm{Out}(\pi_1(M)) \to \mathrm{Out}(\pi_1(M)/\langle A_1 * \cdots * A_n \rangle) \approx \mathrm{Out}(F_m),$$

where $\langle A_1 * \cdots * A_n \rangle$ denotes the normal closure of $A_1 * \cdots * A_n$ in A. Since Γ is a higher-rank, nonuniform lattice, we know from [1] that any homomorphism of Γ into $\mathrm{Out}(F_n)$ has finite image. The claim follows.

Now by hypothesis $\psi_*(\Gamma)$ is infinite. Since $\psi_* = v_1 \oplus v_2$ and $v_2(\Gamma)$ is finite, we have that $v_1(\Gamma)$ is infinite. Hence $\eta_j(\Gamma)$ is infinite for some j. Note that for any j for which η_j is infinite, the representation ξ_{i_j} is infinite and hence M_{i_j} is homeomorphic to T^3.

By re-indexing we may assume $\eta_1(\Gamma_1)$ is infinite. Since the representation

$$\eta_1 : \Gamma_1 \to \mathrm{Aut}(H_1(M_{i_1}; \mathbf{Q})) \approx \mathrm{GL}(3, \mathbf{Z})$$

is infinite, we conclude by exactly the same argument as used in the proof of Theorem I that Γ_1 is a finite-index subgroup of SL(3, **Z**). Hence Γ is commensurable with SL(3, **Z**).

Let N be the connected sum of those M_i for which ξ_i is infinite (and hence $M_i \approx T^3$). Let ρ_1 be the restriction of v_1 to $H_1(N; \mathbf{Q})$. Note that ρ_1 is a direct sum of representations induced from those η_j which have infinite image.

Let α denote the restriction of v_1 to the sum of those $H_1(M_i; \mathbf{Q})$ for which ξ_i has finite image. Now set $\rho_2 = \alpha \oplus v_2$. The conclusion of Theorem II is now clear.

To prove Theorem III, assume that the action of the uniform lattice Γ on M is homologically infinite. The argument that was used to prove Theorem II applies to Γ since the only point in the proof where the nonuniformity of Γ was used was in quoting the result from [1] to show that v_2 has finite image; in the present situation v_2 is a 0-dimensional representation since M has no $S^2 \times S^1$ summands. We conclude that Γ is commensurable with SL(3, **Z**), which contradicts the fact that Γ is uniform.

5. Questions

The following question is a natural complement to Theorem I.

Question. *Let Γ be any finite-index subgroup of SL(n,**Z**), $n > 2$. Is every C^∞, isotopically standard action of Γ on T^n topologically conjugate to an affine action of Γ on T^n?*

The answer to this question is known to be "yes" in the special case of perturbations of the standard, linear action (see [8,14–16,18]). The exotic actions of SL(3, **Z**) on T^3 discussed in Section 3.1 show that a smoothness hypothesis is necessary. Hurder [8] has found examples of affine actions of finite-index subgroups Γ of $\mathrm{SL}(n, \mathbf{Z}), n \geqslant 3$ on T^n which are not conjugate to the standard linear action of Γ. Hence "affine" is the best we can hope for.

Another question that arises is to what extent Theorem II has a converse.

Question. *Let Γ be any finite-index subgroup of $SL(3, \mathbf{Z})$, and let $\rho : \Gamma \to GL(V)$ be a representation on a finite-dimensional vector space V over Q. Suppose that ρ is a direct sum of representations of Γ induced by standard 3-dimensional representations of finite-index subgroups of Γ. Does there exist a closed, connected 3-manifold M with $V \approx H_1(M; \mathbf{Q})$, and an action $\psi : \Gamma \to Homeo(M)$, so that ψ_* is equivalent to ρ?*

The examples produced in Section 3 all have the standard representation as a direct summand. We do not know how to produce an action ψ of $\mathrm{SL}(3, \mathbf{Z})$ on a closed, connected 3-manifold for which ψ_* is a representation of $\mathrm{SL}(3, \mathbf{Z})$ induced by a standard representation of a proper, finite-index subgroup.

Acknowledgements

We thank Geoff Mess for useful comments and corrections.

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